UNDERSTANDING AND ORGANIZING MATHEMATICS EDUCATION AS A DESIGN SCIENCE – ORIGINS AND NEW DEVELOPMENTS[1]

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Abstract

The objective of this paper is

(1) to revisit briefly the conception of mathematics education as a design science as it has been evolving alongside developmental research in the project "Mathe 2000" from 1987 to 2012

(2) to report in some detail on recent developments in the follow-up project "Mathe 2000+," as concerns both conceptual and practical issues.

The paper is a plea for appreciating and (re-)installing "well-understood mathematics" as the natural foundation for teaching and learning mathematics.

Key words: Design science, learning environments, well-understood mathematics, structure-genetic didactical analyses, productive practice, collective teaching experiments

1. ORIGINS

It is no accident that in the 1960s, when the traditional content and methods of teaching mathematics were being questioned and there was a call for new content and methods, the very discipline that had been responsible for the teaching of mathematics for centuries, namely mathematics education (didactics of mathematics), was also questioned by a growing number of mathematics educators (didacticians) and considered to be no longer adequate.

Since at that time, many mathematicians, among them prominent ones, were also committed to mathematics teaching, the discrepancy between the solid scientific foundation of mathematics and its missing counterpart in mathematics education was felt as painful, particularly by those, including the present author, who had moved from mathematics to mathematics education in order to specialize in this rapidly developing field.

In the subsequent discussions about the scientific status of mathematics education, the following questions were foremost:

(1) What is the relationship between mathematics and mathematics education?

(2) What distinguishes mathematics education from mathematics?

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1 The paper is an elaboration of a plenary lecture presented at the 2017 JASME conference in Hiroshima. The author would like to use this opportunity to thank his esteemed Japanese colleagues for a most fruitful personal exchange over almost two decades. In addition, he gratefully acknowledges the valuable information he has been receiving about Japanese mathematics education from his American colleague Jerry P. Becker since 1976. The author would also like to thank him and Yamamoto Shinya for helpful comments on a draft of this paper.

(3) How can mathematics education establish a scientific basis that preserves its close and necessary connections with mathematics and that at the same time reflects its special mission with respect to teaching practice and to teacher education?

At a conference organized by the Institute of Didactics of Mathematics (IDM) at the University of Bielefeld in 1975, Jeremy Kilpatrick made a crucial point when he distinguished between "theories imported from other disciplines" and "theories developed within mathematics education," or "homegrown theories," as he referred to the latter.

Since the early 1970s, the present author has been convinced that mathematics education would be far better served by "homegrown" theories, and so he has been looking for a framework appropriate for developing such theories. His idea to conceive of mathematics education as a design science was inspired by new developments in other fields and in mathematics education itself as will briefly be described in this section.[2]

1.1 The Rise of the Sciences of the Artificial

In 1970, Herbert A. Simon, who in 1978 was awarded the Nobel Prize in economics, published a booklet in which he coined the term "design science" (Simon, 1970). His intention was to delineate disciplines in which he was active (economics, administration, computer science, cognitive psychology) and disciplines like engineering from the established sciences and to provide these disciplines with a scientific *status of their own.* He identified the difference by highlighting "design" as the "principal mark" of design sciences (Simon, 1970, 55):

> Historically and traditionally, it has been the task of the science disciplines to teach about natural things: how they are and how they work. It has been the task of engineering schools to teach about artificial things: how to make artifacts that have desired properties and how to design (…) Design, so construed, is the core of all professional training; it is the principal mark that distinguishes the professions from the sciences. Schools of engineering, as well as schools of architecture, business, education, law and medicine, are all centrally concerned with the process of design.

As "education" was mentioned explicitly here, it was only natural to consider mathematics education as a "design science."[3] The question, however, remained what the "artificial objects" of mathematics education might be. In Wittmann (1984), a proposal was made to consider "teaching units" as these artificial objects. Later this term was replaced by "learning environments."[4]

2 Japanese readers will find a comprehensive description of the main ideas in: 山本信也 (2012). 生命論的デザイン科学として数学教育学の課題と展望 [Prospect and Tasks of Mathematics Education as a Systemic-evolutionary Design Science] 熊日情報文化センター

3 The proposal to understand mathematics education as a kind of "engineering" (in German: "Ingenieurwissenschaft") was first made in Wittmann (1974).

4 In a talk at the University of Dortmund in 1997, Lieven Verschaffel, University of Leuven/Belgium, used the term "teaching/learning unit," which inspired us on the spot to coin the term "learning environment" in Mathe 2000, which was quickly adopted by the community.

1.2 Developments in Management Theory

It is clear that there is a basic difference between a technical artifact, like a machine, which functions according to natural laws, and a "teaching unit" that cannot be used mechanically but requires the intelligent application by human beings as well as adaptation to the momentary social context. This difference is not restricted to education but is also typical of other design sciences, in particular economics.

In 1976, the Swiss management theorist Malik published a book in which he distinguished two classes of design sciences (Malik, 1986):

– *Mechanistic- technomorph* design sciences based on the natural sciences
– *Systemic-evolutionary* design sciences dealing with complex systems that, in contrast with a machine, cannot be completely controlled from outside.

It was equally clear that mathematics education as a design science belongs to the latter class.

1.3 Prototypes of Design in Mathematics Education

The discussion about mathematics education in the 1970s was also very much influenced by fresh contributions to mathematics education that transcended the traditional scope.

In the preface of his book "Basic Notions in Algebra," the eminent Russian mathematician Igor Shafarevics (1989, 4) states:

> The meaning of a mathematical notion is by no means confined to its formal definition; in fact, it may be rather better expressed by a (generally fairly small) sample of the basic examples, which serve the mathematician as the motivation and the substantive definition, and at the same time as the real meaning of the notion.

In the same sense, typical projects in developmental research explain the notion of mathematics education as a design science much better than general descriptions.

In 1965 and 1967, two groups of English mathematics educators published books that consisted of the description of teaching ideas and teaching units combined with the explanation of the mathematical background as well as hints for teaching (Fletcher, 1965; ATM, 1967).

The same approach was pursued on a larger scale at the Dutch Instituut voor Ontwikkeling Wiskunde Oderwijs (IOWO), founded in 1971 under the direction of Hans Freudenthal. A good summary of the developmental research conducted at the IOWO is provided by Freudenthal et al. (1976).

A third important impetus came from Japan. Here again, the progress in mathematics education was communicated by means of carefully formulated teaching units (Becker & Shimada 1997).

Last not least, the work of Heinrich Winter, the "German Freudenthal", must be mentioned as a major inspiration. In Winter's seminal paper on general objectives of mathematics teaching, these objectives were illustrated by teaching examples (Winter 1975).

1.4 The Map of Mathematics as a Design Science

In Wittmann (1995, 89), the conception of mathematics education as a design science was summarized in a diagram that is shown with some modifications in Figure 1.

The *core* of mathematics education (didactics of mathematics) represents the design, the empirical

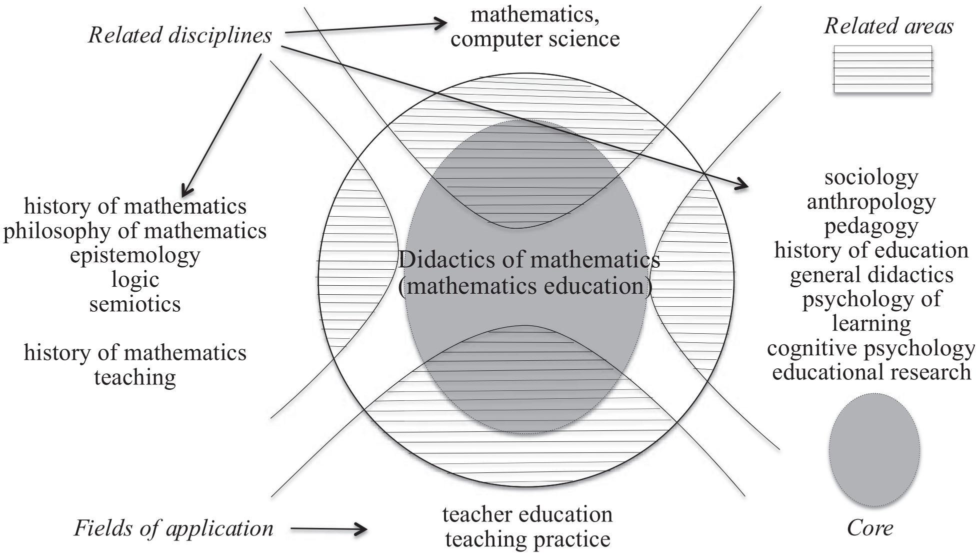

Fig. 1 Mathematics education as a design science and its interdisciplinary relationships

research and the implementation of learning environments. It is surrounded by "*related disciplines*" and "*fields of application.*" The *related areas* are the intersections of mathematics education with the related disciplines.

In Wittmann (2000), this conception was elaborated on with respect to systemic constraints.

The map in Figure 1 was intended to give a foundation to the developmental research undertaken in the projects mentioned in subsection 1.3. However, as it has turned out, this map can also be interpreted differently. As the related disciplines on the right side of the diagram have offered and continue to offer advanced theories of teaching and learning as well as theories on the educational system's societal background, it has been and continues to be tempting for mathematics educators to take these theories as starting points for establishing a scientific basis of mathematics teaching. In the decades since, the mainstream of mathematics education has been moving in this direction to the extent that this approach now enjoys a near monopoly. As this movement has more or less been ignoring the tradition of mathematics education in a similar way in which "New Math" has ignored the tradition of mathematics teaching, the present author suggests calling it "New Math Education." Although "New Math Education" has widened and refined the scope of research and led to progress in many fields, it is clear that it has weakened the connections of mathematics education with both mathematics and teaching practice.

In recent years, an attempt has been made to include "design" in "New Math Education". The result has been termed "design research" (Cobb et al., 2003; Prediger et al., 2015). However, this direction of research, which follows the paradigms of applied science, differs from what was originally intended by conceiving of mathematics education as a design science.

In order to make the difference explicit, it is necessary to highlight the singular role of mathematics among the related disciplines in Fig. 1.

2. CONCEPTUAL DEVELOPMENTS

The conclusion of Mathe 2000 in 2012 was the right moment to rethink the concept on which this project had been based and to formulate a revised concept for the new project Mathe 2000+. It turned out that it was precisely those questions that had been addressed in the discussion about the scientific status of mathematics education in the 1970s that had to be taken up again (see subsection 1.1 above). The answers at which we have arrived will be described in this section. In section 3, some practical consequences will be illustrated by means of typical examples from the project.

2.1 The Natural Theory of Teaching: "Well-Understood Mathematics"

There is no disagreement that mathematics provides the subject matter of teaching and that therefore teachers must "know mathematics" in order to teach the subject properly. A closer look at the problem, however, reveals that there are quite different interpretations of this general statement and that, as a consequence, there are quite different views of the roles that mathematics should play in mathematics education and in teacher education.

At the 1975 Bielefeld conference, John LeBlanc described the basic issue in full clarity (LeBlanc, 1975):

> The content of many mathematics courses was felt to be irrelevant to many of the prospective teachers. The new requirements for the preparation of elementary teachers left mathematics departments looking for materials appropriate for such courses. At the same time, the mathematicians selecting the books were also under some pressure to make sure that the content was mathematically honest. Few, if any, materials existed that met both criteria of educational appropriateness and of mathematical honesty. The latter requirement usually was the winning criterion. The effect of inappropriate but mathematically honest materials was often just the opposite of that which was desired. The prospective teachers seemed to be even less confident than ever in mathematics and their attitude toward it became increasingly negative.

This description might well apply to the present situation in many parts of the world, particularly in the U.S. The "math wars" in this country were definitely driven by different views held by mathematics educators and mathematicians. The books by Jensen (2003) and Wu (2011), both published by the American Mathematical Society, represent the intention to be mathematically honest and suggest that this kind of mathematics is not only necessary but also sufficient for teaching mathematics properly at the elementary level.

However, as early as in 1986, this view was fundamentally challenged by Lee Shulman in a seminal paper in which he contrasted mere "content knowledge" with "pedagogical content knowledge" and "curricular knowledge" (Shulman, 1986). His proposal to look at content in a comprehensive way was taken up and elaborated on in mathematics education in a series of papers (see, for example, Ball et al., 2008).

In the European and Asian contexts, this broader view on content has always been present in teacher education, including teacher education for the elementary level. So it was only in the context of the U.S. that the book by Liping Ma (1999) could be presented as a revelation.

Our attempts in the Mathe 2000+ project to better understand the impact of mathematics on mathematics education were greatly influenced by three papers that John Dewey, one of the greatest minds of all time in the area of education, had published as early as 1903 – 1904.

In Dewey (1903a, 285), an important distinction is made between two different views of a subject:

> Every study or subject has two aspects: one for the scientist as a scientist, the other for the teacher as a teacher (...) For the scientist the subject-matter represents simply a given body of truth to be employed in locating new problems, instituting new researches, and carrying them through to a verified outcome. To him the subject-matter of the science is self-contained (...) He is not, as a scientist, called upon to travel outside its particular bonds. (...) The problem of the teacher is a different one. As a teacher he is not concerned with adding new facts to the science he teaches. (...) He is not concerned with the subject-matter as such, but with the subject matter as a related factor in a total and growing experience. Thus to see it is to psychologize it.

The connections and the differences between logically and psychologically organized subject matter were clarified in another eye-opening paper of Dewey's in great detail where Dewey arrived at the following conclusion (Dewey, 1903b, 227 - 228):

> The serious problem of instruction in any branch is to acquire the habit of viewing in a twofold way which is taught day by day. It needs to be viewed as a development *out* of the present habits and experiences of emotion, thought, and action; it needs to be viewed also as a development *into* the most orderly intellectual system possible. These two sides, which I venture to term the psychological and the logical, are the limits of a continuous movement rather than opposite forces or even independent elements.

In Dewey (1904), a whole chapter is devoted to the role that courses on the subject matter of teaching should play in teacher education. Dewey insists on seeing subject matter not only as a fixed body of knowledge but also as a developing process that involves methods of teaching as well (Dewey, 1904, 263 – 264):

> Scholastic knowledge is sometimes regarded as if it were something quite irrelevant to method. When this attitude is even unconsciously assumed, method becomes an external attachment to knowledge of subject-matter. It has to be elaborated and acquired in relative independence from subject-matter, and *then* applied.
>
> Now the body of knowledge which constitutes the subject-matter of the student teacher must, by the very nature of the case, be organized subject-matter. It is not a separate miscellaneous heap of scraps. Even if (as in the case of history and literature), it be not technically termed "science," it is none the less material which has been subjected to method – has been selected and arranged with reference to controlling intellectual principles. There is, therefore, method in subject-matter itself – method indeed of the highest order which the human mind has yet evolved, scientific method.
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> It cannot be too strongly emphasized that this scientific method is the method of the mind itself. The classifications, interpretations, explanations, and generalizations which make subject-matter a branch of study do not lie externally in facts apart from mind. They reflect the attitudes and workings of mind in

> its endeavor to bring raw material of experience to a point where it at once satisfies and stimulates the needs of active thought. Such being the case, there is something wrong with the "academic" side of professional training, if by means of it the student does not get constantly get object-lessons of the finest type in the kind of mental activity which characterizes mental growth and, hence, the educative process. (...) Only a teacher thoroughly trained in the higher levels of intellectual method and who thus has constantly in his own mind a sense of what adequate and genuine intellectual activity means, will be likely, in deed, not in mere word, to respect to the mental integrity and force of children.

Shulman (1986, 6 - 7) elaborates on the fact that in antiquity and in the Middle Ages there was no distinction between research and teaching. It is no accident that the term "mathematics" is derived from the Greek μαθηματική τέχνη (mathematike techne), which denotes the "art of teaching and learning." Even in modern Greek, μαθαίνω (mathaino) means "learning."

Later on, "knowledge" and "learning" became more and more separated. However, at the forefront of research, the connection between research and teaching has always been close, right up to the present day. William Thurston, who was awarded the Fields Medal in 1982, defended his use of broader means of representation with his intention to support understanding (Thurston 1994, 162):

> It may sound almost circular to say that what mathematicians are accomplishing is to advance human understanding of mathematics (...) If what we are doing is constructing better ways of thinking, then psychological and social dimensions are essential to a good model for mathematical progress.

From these descriptions we have drawn the following conclusion: If mathematics is understood and practiced as a living and developing organism including guiding problems, heuristic strategies for solving problems, different types of representation (enactive, iconic, symbolic), different ways of communication, exercises at different levels, the search for structures and patterns, proofs, and applications to internal or real-world problems, then the organization of knowledge in order to make it understandable to students is part and parcel of this "well-understood mathematics" *at any level,* beginning with early math (Kinnear & Wittmann, 2018).

Mathematics has grown historically, and this growth provides a good starting point for appreciating "well-understood mathematics." This is not to say that teaching should follow the historical order. However, history gives valuable information about how to develop mathematics *genetically* and shows that the lower stages of development are *indispensable* for the higher stages and must be appreciated in their *specific manifestations*. It is a fundamental mistake to believe that formal mathematical analyses of the subject matter can replace elementary formulations (see Dewey 1903b). "Mathematical honesty" must not be reserved for formal analyses, which nevertheless are indispensable as an orientation for the development of coherent curricula (see section 2.2).[5]

If mathematics education is based on "well-understood mathematics," then there is no compelling reason to look *exclusively* for theories of teaching and learning outside of mathematics in order to secure a scientific basis. *The natural theory of teaching and learning mathematics is implicit in "well-understood*

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5 In the eyes of the present author, U.S. mathematicians lost the "math war" because of their inappropriate top-down perspective. It is a pity that they failed to refer to "well-understood mathematics."

mathematics." The literature on elementary mathematics offers a veritable goldmine for "well-understood mathematics" waiting to be exploited in the design of learning environments and in teacher education. Research on the history of mathematics and on the philosophy of mathematics is of great help in elaborating on this natural theory.

2.2 Structure-Genetic Didactical Analyses

In traditional mathematics education, didactical analyses have been the main method for shaping conceptions for teaching certain areas. This method is taken up in the design science approach but further specified in the following way: Subject matter is considered *in its development* with respect to the development of learners at different levels. Both "mathematical honesty" and "educational appropriateness" are taken seriously and brought to a natural synthesis. In order to express this extended method properly, the term "structure-genetic didactical analyses" has been coined (Wittmann 2018).

Structure-genetic didactical analyses represent a "bottom-up" view of teaching and learning that fundamentally differs from a "top-down" view. In this respect, Freudenthal's critique of Chevallard's demand for a "didactical transposition" from the knowledge of scholars to the school environment is enlightening (Freudenthal 1986, 326 - 327).

Structure-genetic didactical analyses offer important advantages (Wittmann 2018, 145):

1 They emerge from mathematical practice, that is, from doing mathematics at various levels.
2. They foster an active relationship with mathematics.
3. They are constructive and therefore absolutely essential for designing substantial learning environments and coherent curricula.
4. They are natural guidelines for teachers, as they bring to fruition the implicit theories of teaching and learning mathematics and "unfreeze the didactical moments frozen in the subject" (Heintel 1978, 46).

Point 3 is most important, as success in learning greatly depends on linking new knowledge to old knowledge in a coherent way. This is in line with David Ausubel's famous statement (Ausubel, 1985):

> The most important single factor influencing learning is what the learner already knows. Ascertain this and teach him accordingly.

What the learner knows, however, is mainly determined by previous learning. Therefore, a coherent and consistent curriculum in which care is taken to establish solid knowledge *continuously* plays a decisive role in teaching and learning. In order to design such curricula a thorough knowledge of mathematics *across all levels,* including logical analyses, is crucial.

2.3 A Differentiated Conception of Practicing Skills

While subsections 2.1 and 2.2 are concerned with connecting mathematics education to mathematics, that is, to "well-understood mathematics," this subsection and the next one are devoted to linking mathematics education to teacher education and teaching practice.

One of the oldest principles of learning is summarized in the Latin saying, "Repetitio mater studiorum" (practice makes perfect). While this principle is unquestioned in fields like music or sports, it is often woefully neglected when it comes to learning. It is indicative that in contemporary mathematics education, notably in "New Math Education," "practice" is hardly a topic of research. In the Western world, "practice" is most commonly understood as "drill and practice" and therefore rejected in principle, in marked contrast with Asia. However, even teachers in Western countries know that without extensive "practice," no real and lasting progress is possible.

The Mathe 2000 project made deliberate attempts to overcome the seeming contradiction between "practice" and "understanding" or between "genuine mathematics" and the "basics." The result was a differentiated conception of practicing skills. In Wittmann & Müller (2017, 141) a distinction is made between three types of practice:

"Introductory practice" aims at making students familiar with a new topic, that is with new problems, new means of representation, new vocabulary, new symbols, new methods, etc. The main objective is to firmly link new knowledge to old knowledge. According to Wittgenstein's language game, students acquire new knowledge through its *repeated* use in meaningful contexts.

"Basic practice" refers to the extended practice of a small set of skills that occur frequently, the so-called *basic competences*, which must be mastered *automatically.*

"Productive practice" is a kind of magic wand: It integrates the practice of skills with the exploration and explanation of patterns, with the solution of problems and with applications. The term *"productive practice"* was coined by Heinrich Winter, who wanted to emphasize that students are expected to "produce" something on their own in this type of practice. *Productive practice* represents the view of mathematics as the science of patterns in full manifestation, and, at the same time, it offers a number of practical advantages (little preparation on the part of teachers, self-monitoring on the part of students, natural differentiation between students, time saved).

2.4 Awareness of Systemic Constraints

It is a tacit assumption in "New Math Education" that teaching and learning represent realities that can be investigated and controlled roughly in the same way as physical realities. This assumption is a mere fiction. Of course, there are *momentary* local realities of teaching and learning. However, these are not given and enduring but rather shaped by former (formal or informal) teaching and learning, however this may have taken place, and they are fluid and shaky. Donald Schön has convincingly shown that in complex systems, the methods of "applied science" are of limited value and that therefore the professionals in these fields must make decisions on their own in their local environment (for more details see Wittmann 2016).

As it is the communication with teachers that counts, it has been a conscious decision in our project to provide them with a robust theory for teaching that can be communicated in understandable language and to empower teachers to act in a self-reliant way. Here again, the design science approach and "New Math Education" differ fundamentally. The results of research in "New Math Education" are extremely diverse and formulated in a technical language. Their sheer mass is excessive, and it is hard to imagine how they can reach the practice of teaching.

The most promising way for taking systemic constraints into account seems to be introducing teachers to "well-understood mathematics" and connecting it to substantial learning environments. In our view, it is first-hand knowledge in this area that is the most important asset in terms of professional knowledge. This knowledge greatly facilitates work with students, including with respect to communication and social interaction, and it provides teachers with the necessary flexibility in subject matter that is needed for meeting the demands of individual students.

The natural theory of teaching and learning as implicit in "well-understood mathematics" might appear "naïve" in comparison to the grand theories offered by "New Math education." However, from the systemic perspective this is a decisive advantage.

From this perspective, there is another important point: Although international exchange in mathematics education has made huge progress compared to the pre-1970s situation, its impact on schools can only be effective at a *local* level. In this respect, the involvement of teachers is crucial (Fung 2016).

3. PRACTICAL CONSEQUENCES

In this section, some practical consequences of the new conceptual basis will be illustrated by means of examples that are taken from the new "Handbook of Practicing Skills in a Productive Way" (Wittmann & Müller, 2017/2018). The choice of title was a deliberate decision with respect to our systemic credo. In fact, this handbook is not just about practice but rather offers a comprehensive introduction to teaching arithmetic in the first four years.

The following four subsections are only loosely linked to the subsections in section 2 as the various points discussed in this section overlap and, as a rule, each learning environment exemplifies several of them.

3.1 Integrating "Well-Understood Mathematics"

In a letter submitted to the working group on proof at ICME 7, Québec 1992, Yuri I. Manin introduced the term "mathscape" for the mathematical landscape research mathematicians see in their mind's eye and explore.[6] It is only natural to combine this metaphor with the term "learning environment" and, in a further step, to compare the role of the teacher with the role of a mountain guide. The job of the latter is to select tours that are appropriate for a certain group of hikers and to guide them to certain summits. In order to act professionally, the guide must have *first-hand experience* of mountainous landscapes, know how demanding certain tours are and have options for changing or abridging tours it if this should be advisable or even necessary. In a similar way, teachers must have first-hand experience of the "mathscape" on which a learning environment is based, and they must have different hiking options at their disposal.

In the new handbook, the subject matter is ordered in chapters devoted to subject areas. For each area, several learning environments are offered. Each chapter starts with an introduction into the mathematical structure of the area in question. The description of each individual learning environment also begins by laying out the mathematical structure of this environment in more detail. Teachers are invited to *first* "go on tour" *for themselves* in order to become familiar with the "mathscape" involved.

Example: The Learning Environment "Guessing Dice"

This environment is part of the subject area "Productive practice of addition and subtraction" in grade 1 (Wittmann & Müller, 2017, 125 – 127).

The guiding problem is as follows: The teacher rolls three dice behind a barrier and announces only the

6 Manin's letter is reprinted in Wittmann 2002, 546.

total of the three numbers. In order to find out which numbers were rolled, the children are allowed to ask questions which can be answered with "yes" or "no," for example, *"Is there a 6?"* or *"Is there a 4?"*

Any answer provides additional information so that the numbers can be determined step by step.

Like all environments for *productive practice,* this learning environment requires a certain mastery of the skills that are involved – in this case, the addition and subtraction tables. However, these skills are applied in various ways and thus corroborated and consolidated.

For some totals (for example, 3, 4, 17, 18), no questions are needed, as there is only one triple of numbers with this total. For other totals there are up to six triples of numbers. Table 1 shows the combinatorial possibilities. These are called *partitions,* as the order of the summands does not matter. In order to avoid multiplicity, it makes sense to write the three summands in decreasing order. Listing the partitions in lexicographic order is a way to determine them systematically. This method can also be applied in order to determine partitions for which the size of the biggest part is not restricted, unlike in this case, where the limit is 6.[7]

Table 1 Partitions of the numbers 3 to 18 in three parts ≤ 6

Total	Decompositions
3	1+1+1
4	2+1+1
5	3+1+1, 2+2+1
6	4+1+1, 3+2+1, 2+2+2
7	5+1+1, 4+2+1, 3+3+1, 3+2+2
8	6+1+1, 5+2+1, 4+3+1, 4+2+2, 3+3+2
9	6+2+1, 5+3+1, 5+2+2, 4+4+1, 4+3+2, 3+3+3
10	6+3+1, 6+2+2, 5+4+1, 5+3+2, 4+4+2, 4+3+3
11	6+4+1, 6+3+2, 5+5+1, 5+4+2, 5+3+3, 4+4+3
12	6+5+1, 6+4+2, 6+3+3, 5+5+2, 5+4+3, 4+4+4
13	6+6+1, 6+5+2, 6+4+3, 5+5+3, 5+4+4
14	6+6+2, 6+5+3, 6+4+4, 5+5+4
15	6+6+3, 6+5+4, 5+5+5
16	6+6+4, 6+5+5
17	6+6+5
18	6+6+6

Teachers who are familiar with the structure in Table 1 will be well prepared for guiding children through this learning environment.

In order to activate the readers of the handbook beyond the mathematics in the learning environments, each chapter ends with a section called "Search and Find for the Reader." The problems that are offered for investigation here use only the mathematics and means of representation of this chapter and so contribute to enhancing the professional knowledge of teachers. Although the problems are somewhat more demanding, they are nevertheless accessible experimentally and can also be tackled by talented students.

Example:

7 *Partitions* are well suited as a field of study in mathematical courses for teachers.

In the chapter "Introduction to the Thousand Space," digit cards for the numbers 0, 1, 2, 3, 4, 5, 6, 7, 8, 9 are used as one of the standard teaching aids. The problem in the corresponding section, "Search and Find for the Reader," is as follows:

Take the nine digit cards

1	2	3	4	5	6	7	8	9

and arrange them into three 3-digit numbers so that

a) the difference between the biggest and the smallest number is as big as possible,

b) the difference between the biggest and the smallest number is as small as possible,

c) the difference between the biggest and the middle number and the difference between the middle number and the smallest number are equal.

d) Try to make the difference in c) as small as possible.

Hint: The smallest possible difference is smaller than 100.

3.2 Designing a Consistent and Coherent Curriculum

In the revision of the handbook, structure-genetic didactical analyses were used both globally and locally.

Local analyses were applied for designing "mathscapes" that invite students (and teachers) to mathematical activities (see the examples in 3.1 and 3.3).

The dominant *global* objective was to design a consistent and coherent curriculum. "Consistent" means that the language, the means of representation and the problems to be addressed should fit together over the grades. "Coherent" means that there should be a seamless sequence of learning environments that build on one another.

The main instrument for achieving curricular consistency and coherence was the following list of seven fundamental ideas of arithmetic that allow for a genetic development of the subject matter (Wittmann & Müller 2017, 144):

1. Number as a synthesis of the ordinal and the cardinal aspect
2. Arithmetical laws
3. The structure of the decimal system
4. Algorithms
5. Arithmetical patterns
6. Numbers in the environment
7. Applications

Special attention was paid to the idea in No. 2, as Heinrich Winter's demand for an "algebraic penetration of arithmetic" should be put into practice. To this end, the arithmetical laws had to be introduced *as early as possible*. Addition and subtraction presented no problem as the associative and the commutative laws of addition can easily be based on operations with counters. For multiplication and division, we chose rectangular arrays of counters and dots for the simple reason that this representation is the *only one* which allows for establishing the associative and commutative laws of multiplication as well as the distributive law at an elementary level (Freudenthal, 1983, 109).

In Wittmann & Müller (2017, 71, 202 – 204), operative proofs of the five laws are presented that rest on the following invariance principle: The cardinal number of a set of counters (or dots) is independent of the location of the counters (dots).

The proofs run as follows:

Addition means that two sets of counters are united to form one set. Whether this operation is executed in one or several steps does not affect the result. In algebraic formulation: $a + (b + c) = (a + b) + c$.

Also the result does not depend on the order in which the two sets are put together: $a + b = b + a$.

The commutative law of multiplication is easily derived from the fact that rows and columns in a rectangular array $a \cdot b$ of dots change roles when the array is rotated by 90°. No dot is taken away, no dot is added. Therefore $a \cdot b = b \cdot a$.

Any array $a \cdot b$ can be separated into two arrays by means of a vertical or a horizontal segment or into four arrays by means of a horizontal and a vertical segment. In this way, the distributive law is established:

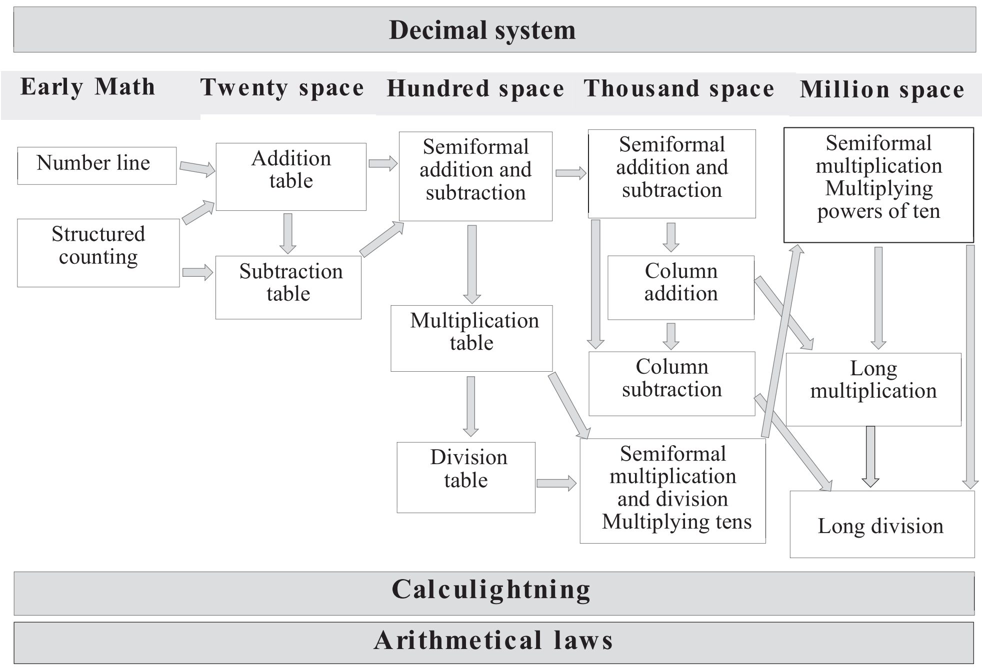
$a \cdot (b_1 + b_2) = a \cdot b_1 + a \cdot b_2$, $(a_1 + a_2) \cdot b = a_1 \cdot b + a_2 \cdot b$

$(a_1 + a_2) \cdot (b_1 + b_2) = a_1 \cdot b_1 + a_1 \cdot b_2 + a_2 \cdot b_1 + a_2 \cdot b_2$

If we take an array $a \cdot b$ and arrange c copies of it consecutively, we get a large array with $c \cdot (a \cdot b) = (a \cdot b) \cdot c$ dots. As this array has a rows and $c \cdot b = b \cdot c$ dots in each row, we get $(a \cdot b) \cdot c = a \cdot (b \cdot c)$.

Table 2 gives an overview of the curricular structure of arithmetic for grades 1 to 4 at which we have arrived. The coherence of the topic areas is expressed with arrows.

Table 2 The curricular structure of arithmetic in grades 1 to 4



The table contains two bars that indicate the continuous development of the two fundamental ideas "Decimal system" and "Arithmetical laws" over the grades.

In addition, there is a third bar: “Calculightning” (in German “Blitzrechnen”). This word is an artificial combination of the words “calculating” and “lightning” and denotes a course with 40 basic competences that have to be mastered so they can be performed automatically (see Table 3).[8]

As the names of the modules indicate, these competences are not independent but build on one another, both horizontally across the number spaces and vertically within each number space.

Table 3 Overview of the course „Calculightning“

Twenty Space	Hundred Space	Thousand Space	Million Space
How many?	How many? Which number?	Multiplication and division table	Reading and writing big numbers
Row of twenty	Counting in steps	Doubling/Halving in the hundred space	Complementing to 1 million
Power of Five	Complementing to the next ten	How many? Which number?	Dividing powers of 10 in equal parts
Decomposing	Complementing to 100	Counting in steps	Subtracting powers of ten
Complementing to 10/20	Dividing 100 in equal parts	Complementing to 1000	Reading numbers differently
Doubling	Doubling/Halving	Dividing 1000 in equal parts	Counting in steps
Addition table	Easy addition problems	Doubling/Halving in the thousand space	Doubling/Halving in the million space
Subtraction table	Easy subtraction problems	Easy addition and subtraction problems	Easy addition and subtraction problems
Halving	Decomposing	Multiplying by 10/ Dividing by 10	Multiplying powers of 10
Counting in steps Mini-times table	Multiplication table	Multiplying and dividing tens	Easy multiplication and division problems

“Calculightning” is important for two reasons: It secures not only a firm mastery of a small set of basic competences, but it also serves as a remedial program for students who need additional support. At first glance, these two objectives might appear contradictory. However, a closer look at “Calculightning” reveals that this course contains exercises that are crucial for understanding the decimal system and for establishing connections between number facts. Moreover, all modules of “Calculightning” are introduced in the context of *introductory practice,* which aims at facilitating understanding. Automation then comes in as the very final step in mastering these basic competences.

For the multiplication table this means:

This table is introduced by means of arrays of rectangular dots, whereby the connections provided by the arithmetical laws are used for effective learning.

It is only after extensive *introductory practice* that *basic practice* of the table begins with the ultimate goal of automation.

Learning environments for *productive practice* are investigated in parallel to this and include operative

8 “Calculightning” (“Blitzrechnen”) is available in the form of four apps (corresponding to grades 1 to 4) that offer also an option for English.

proofs based on rectangular arrays of dots (for details see Wittmann & Müller 2017, chapters H 4, 5, and 6).

3.3 Including Operative Proofs

Proofs are "the very heart of mathematics" (Günter Ziegler). At lower levels, it is appropriate to include "operative proofs" that use mathematical structures integral to informal means of representation (Wittmann 2002, 545 – 548). The conception of *productive practice* is well suited to combine the practice of skills with mathematical investigations, including proofs, and thus reflects "well-understood mathematics" in a particularly significant way.

A typical example is provided by the learning environment "ANNA numbers" (Wittmann & Müller 2018, section M 2.2.3). In this environment, the practice of column subtraction is combined with a mathematical investigation.

ANNA numbers are 4-digit numbers like 4114, 7887, 3003, etc. For any pair of different digits, there are two ANNA numbers, and the smaller number of each pair can be subtracted from the larger one: 4114 – 1441, 8778 – 7887, 3003 – 0330, etc.

During the calculations, the following patterns emerge and can be discovered by students (and teachers):

(a) Only the results 891, 1782, 2673, 3564, 4455, 5346, 6237, 7128, and 8019 are possible, and they show conspicuous patterns.

(b) ANNA numbers with the same difference of digits have the same result.

(c) All results are multiples of the smallest result 891.

One proof of these patterns uses the place value table (Figure 2) and runs as follows:

Usually subtraction is defined as "taking away." A second, mathematically more advanced interpretation is "complementing." Here a – b means finding the number c which added to b yields a. In other words: b is complemented by c to yield a, and c is the difference a – b.

The left place value table in Figure 2 shows how the number 1221 is complemented to equal 2112. One counter is moved from the tens column to the ones column, and one counter from the hundreds column is moved to the thousands column. This operation increases the number 1221 by + 1000 + 1 – 100 – 10 = 891.

Fig. 2 Operative proof based on the place value table

In the place value table on the right side of Figure 2, two pairs of counters must be moved correspondingly. Therefore, the difference between 3113 and 1331 is 2 · 891 = 1782.

In order to go from 4994 to 9449, five pairs of counters must be moved. The difference is 5 · 891 = 4455. Clearly, the difference of the digits determines how many pairs of counters must be moved.

A second proof uses the semiformal strategy "separate place values" that is introduced as early as grade

2 with 2-digit numbers (Figure 3).[9]

7447 – 4774 = 2673	5225 – 2552 = 2673	8228 – 2882 = 5346	6006 – 0660 = 5346
7000 – 4000 = 3000	5000 – 2000 = 3000	8000 – 2000 = 6000	6000 – 0 = 6000
400 – 700 = – 300	200 – 500 = – 300	200 – 800 = – 600	0 – 600 = – 600
40 – 70 = – 30	20 – 50 = – 30	20 – 80 = – 60	0 – 60 = – 60
7 – 4 = 3	5 – 2 = 3	8 – 2 = 6	6 – 0 = 6

Fig. 3 Operative proof based on the semiformal strategy „separate place values“

The calculations show that the results depend only on the difference of the digits and that all results are multiples of the smallest result 1000 – 100 – 10 + 1 = 891.[10]

A third operative proof starts with the fact that a difference remains unchanged if both the minuend and the subtrahend are increased by the same number.

Starting from 1001 – 0110 = 891 and increasing both digits by 1 step by step increases both numbers by 1111 and leads to 2112 – 1221, 3223 – 2332, 4334 – 3443, etc. All these differences have the same result of 891.

Starting from 2002 – 0220 = 1782 and increasing the digits by 1 step by step leads to 3223 – 2332, 4334 – 3443, etc. Again, the result 1782 does not change.

In an analogous way, we can start with 3003 – 0330 = 2673 or 4004 – 0440 = 3564 etc.

In all cases, the results remain invariant.

In order to show that all results are multiples of 891, we start from 1001 – 0110 = 891 and pass over to 2002 – 0220, 3003 – 0330, …, 9009 – 0990. At each step, the minuend increases by 1000 + 1, the subtrahend by 100 + 10. Therefore the difference grows by 1000 + 1 – 100 – 10 = 891. So 1728 = 891 + 891, 2637 = 1782 + 891, etc.

A fourth operative proof, which is preferable in the context of practicing column subtraction, rests on an analysis of the subtraction algorithm. In Germany the "complementing method" is still widespread (although unfortunately the mathematically less advanced method common in English-speaking countries is gaining ground). Figure 4 shows some calculations according to the "complementing method." As the name indicates, this method consists of complementing the subtrahend such that the subtrahend and the complement add up to the minuend. The notation is minimalistic, and the connection with column addition is obvious.

1001	4334	5335	6446	5225	6336
– 0110	– 3443	– 3553	– 4664	– 2552	– 3663
1 1	1 1	1 1	1 1	1 1	1 1
891	891	1782	1782	2673	2673

Fig. 4 Operative proof based on column subtraction (complementing method)

9 The calculations run as follows: First the problem is written down. Then a line is drawn, and under it, the subtractions for the different place values are executed. Finally, the partial results are mentally combined for the final result, which is then entered in the first line.

10 This proof is close to the algebraic proof where a pair of ANNA numbers is represented by A · 1000 + B ·100 + B · 10 + A · 1 and B · 1000 + A · 100 + A · 10 + B · 1, A > B. The difference of the two numbers is (A – B) · (1000 – 100 – 10 + 1) = (A – B) · 891.

In all results of Figure 4, the tens' digit and the ones' digits add up to 10, the hundreds' digit is 1 less than the tens' digit, and the thousands' digit is 1 less than the ones' digit. As a consequence, the ones' digit of the result, which is the difference of the digits of the ANNA number, determines the whole result.

All operative proofs are rigorous as they rest on *generally applicable operations* independently of the examples by which they are demonstrated.

This learning environment is well suited to illustrate three general points:

(1) Teaching aids must be selected carefully, and the decisive criterion is how well they incorporate the mathematical structure and can be used for operative proofs (Wittmann 1988).

(2) Decisions about which methods of calculating are preferable can only be made by looking *at the whole curriculum*. Otherwise, not only will the consistency and the coherence be adversely affected but so will the mathematical impact.

(3) In guiding students through a learning environment, the teacher has mainly to follow the natural flow of the mathematical activity rooted essentially in "well-understood mathematics" as it is perfectly captured in Guy Brousseau's theory of didactical situations: introduction, action, communication, validation, institutionalization (Brousseau 1997).

3.4 Addressing Teachers as "Reflective Practitioners"

The new handbook stimulates readers to become active not only mathematically but also didactically. In this regard, the most important new feature in the handbook is an adaptation of the Japanese lesson study method (see Becker & Shimada 1997; Hirabayashi 2002). Each chapter of the handbook ends with some proposals for conducting "collective teaching experiments." This term was inspired by the French philosopher Bruno Latour, who introduced it in environmental sociology (for details see Wittmann 2016).

Teachers must be made aware that for systemic reasons, researchers cannot collect and communicate all knowledge that is needed for teaching. So teachers must sensibly implement what is proposed to them, adapt it to their experiences and routines, and collect further information in the classroom themselves.[11]

The following is a typical example for a collective teaching experiment: Readers are invited to compare the proposal for guiding students through the learning environment "ANNA numbers" with another approach in which a similar environment about "UHU Numbers" is first explored.[12] It is interesting to observe to which extent students are able to transfer what they have learned about UHU numbers to ANNA numbers.

All collective teaching experiments in the handbook are numbered to facilitate an exchange about students' experiences.

11 Hiro Ninomiya has sensibly pointed to the importance of Japanese teachers' "implicit" knowledge for conducting lessons. The present author has often found that the systemic thinking for which he has to plea fervently in his context is implicit in Japanese education in many ways – so implicit that in Japanese there is not even a word for "systemic."

12 UHU numbers are numbers like 343, 727, etc. "Uhu" is the German name for eagle owl. For any two pairs of different digits, there are two UHU numbers which can be subtracted: 434 – 343, 727 – 272, ... The results 91, 182, 273, ..., 819 also show a conspicuous pattern and are multiples of 91.

FINAL REMARKS

1. "Well-understood mathematics" should not only be integrated into didactical courses, textbooks and materials for teachers. To organize *mathematical courses* in teacher education accordingly would greatly contribute to improving the image of mathematics, both with teachers and student teachers, and to provide them with highly effective professional knowledge. This knowledge could be further developed in didactical courses ("methods courses").

It seems promising to design mathematical courses for primary teachers starting with the mathematics that is integrated in the handbook. To rephrase Dewey: In this way, student teachers would "constantly get object-lessons of the finest type in the kind of mental activity which characterizes mental growth and, hence, the educative process."

2. Although structure-genetic didactical analyses provide a great deal of empirical evidence for teaching, it is worthwhile to conduct controlled teaching experiments and to document the processes that can be observed. Documentation of this kind is very valuable in teacher education (Hirabayashi 2002). Here, young researchers will find ample opportunities.

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